Norton's Trace Formulae for the Griess Algebra of a Vertex Operator Algebra with Larger Symmetry

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Abstract. Formulae expressing the trace of the composition of several (up to five) adjoint actions of elements of the Griess algebra of a vertex operator algebra are derived under certain assumptions on the action of the automorphism group. They coincide, when applied to the moonshine module $V^\natural$, with the trace formulae obtained in a different way by S. Norton, and the spectrum of some idempotents related to 2A, 2B, 3A and 4A element of the Monster is determined by the representation theory of Virasoro algebra at $c = 1/2$, W_3 algebra at $c = 4/5$ or W_4 algebra at $c = 1$. The generalization to the trace function on the whole space is also given for the composition of two adjoint actions, which can be used to compute the McKay-Thompson series for a 2A involution of the Monster.

Introduction

Since Griess' construction of the Monster simple group [Gr1] as the automorphism group of a commutative nonassociative algebra of dimension 196883+1, many attempts are made in order to understand the nature of this algebra. Conway [Co] reconstructed a slightly modified version of the algebra, called the Conway-Griess algebra, and gave a description of a 2A involution (a transposition) in terms of the eigenspace decomposition with respect to the adjoint action of an idempotent of a particular type called the transposition axis. Some more formulae related to the decomposition of this kind are obtained by Norton

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[No]. In particular, he wrote down a trace formula for the composition of several (up to five) adjoint actions of elements of the algebra:

$$\mathrm{Tr}\, ad_a,\ \mathrm{Tr}\, ad_a ad_b, \dots.$$

These results are based on the explicit construction of the algebra as well as the character table of the Monster and some of its subgroups.

On the other hand, Frenkel et al. [FLM1] constructed a graded vector space $V^\natural$ called the moonshine module, and showed that the Conway-Griess algebra is naturally realized as the subspace of degree (conformal weight) 2 of $V^\natural$. The multiplication and the inner product of the algebra are actually a part of an infinite series of bilinear operations on the whole space $V^\natural$ giving this space with the structure of a vertex operator algebra (VOA) [Bo1],[FLM2]. The moonshine module $V^\natural$ with the VOA structure is probably the most natural object to be considered in the study of the Monster in its relation to the moonshine phenomena, as it was more or less clear in the original construction of $V^\natural$ in [FLM1], and further supported by Borcherds's solution [Bo2] to the Conway-Norton conjecture [CN]. The algebra formed by the degree 2 subspace of a VOA is generally called the Griess algebra of the VOA.

Recently an attempt to understand the Conway-Griess algebra from the VOA point of view was made by Miyamoto [Mi1], who opened a way to study the action of Monster elements on the moonshine module by using a subVOA whose fusion rules have a nice symmetry. In particular, 2A involution of the Monster is described as an automorphism obtained by the eigenspace decomposition with respect to the action of Virasoro L_0 corresponding to the subVOA isomorphic to $L(\frac{1}{2}, 0)$, which reproduces, on the level of the Griess algebra, the action described by Conway mentioned above. Further, the structure of $V^\natural$ as a module over the tensor product of 48 copies of $L(\frac{1}{2}, 0)$, called a frame, was studied in [DGH], and the VOA structure of the moonshine module was reconstructed from a frame in [Mi3]. In particular, the character and the McKay-Thompson series for 2A, 2B and 3C elements of the Monster can be computed by using a frame.

The primary purpose of this paper is to derive Norton's trace formula concerning the adjoint action of elements of the Griess algebra mentioned above from the VOA structure on the moonshine module $V^\natural$ without using any explicit structure of the Griess algebra or the Monster; the only particular property necessary for our derivation is the fact that the components of $V^\natural$ fixed by the full automorphism group coincides with the Virasoro submodule generated by the vacuum vector up to degree 11. This property says that, while the automorphism group is finite as it is the Monster, the symmetry of the VOA is large enough to separate the trivial components into the small subspace of which the action on the VOA is determined by the action of the Virasoro algebra (up to that degree).

It should be emphasized that, in our derivation of the formulae, we do not need to know that the automorphism group is the Monster so that our method is generally applicable to any VOA with the same property; We call it a VOA of class $\mathcal{S}^n$ if its trivial components with respect to the full automorphism group coincides with the Virasoro submodule generated by the vacuum vector up to degree n. This is what we mean by a VOA with larger symmetry.

We will actually show in this paper (Theorem 1) that the trace of the composition of m adjoint actions of elements of the Griess algebra, $m = 1, \dots, 5$, is expressed in the same

way as Norton's formula with coefficients being replaced by certain rational functions of the rank (central charge) c and the dimension d of the Griess algebra if the VOA is of class $\mathcal{S}^{2m}$ under some technical assumptions.

In spite that the trace must be invariant under the cyclic permutation of the order of the operators involved in the trace, the explicit expression for $m = 4$ and 5 does not satisfy this property in general; it means that there is a restriction on the pair (c, d) coming from our assumptions. Also, in a slightly different way, we see that if the Griess algebra contains an idempotent with central charge being different from 0 and c then similar restrictions are imposed on the pair (c, d). In this way, we may list up the possible pairs of (c, d) of a VOA satisfying our assumptions (Section 3). In particular, if the VOA is of class $\mathcal{S}^8$ and it has an idempotent as above, then the rank must be 24 and the dimension must be 196884, i.e., those of the moonshine module $V^\natural$ (Theorem 3.2).

Now, as we have established Norton's trace formulae in a different way, we may use them to study the action of some elements of the Monster. Indeed, the trace formulae has sufficient information to determine the spectrum of some idempotents related to 2A, 2B, 3A and 4A element of the Monster by the representation theory of $L(\frac{1}{2}, 0)$, W_3 algebra at $c = 4/5$ or W_4 algebra at $c = 1$ (Subsection 4.2), reproducing some of the results of Conway [Co] and Norton [No] in the opposite way.

We note that the trace formulae would be generalized to the traces on the higher degree subspaces. In fact, we will show (Theorem 5.1) that, under suitable assumptions, the trace functions

$$\operatorname{Tr} a_{(1)} q^{L_0}, \quad \text{and} \quad \operatorname{Tr} a_{(1)} b_{(1)} q^{L_0},$$

where a, b are elements of the Griess algbera, are indeed expressed in terms of the character $\operatorname{ch} V$ and its derivatives with coefficients written by $(a|\omega)$, $(b|\omega)$ and $(a|b)$ as well as the Eisenstein series $E_2(q)$ and $E_4(q)$ by using some identities established by Zhu [Zh]. As a corollary, we show that the McKay-Thompson series $T_{2\mathrm{A}}(q)$ for a 2A involution of the Monster can be computed from $\operatorname{ch} V^\natural = q(J(q) - 744)$ and the characters of $L(\frac{1}{2}, h)$, $h = 0, 1/2, 1/16$, without using a frame.

We finally note that our consideration based on the nonexistence of a Monster invariant primary vector of degree less than 12 in the moonshine module can be understood to be complementary to the result of Dong and Mason [DM] that the one-point function for a Monster invariant primary vector of degree 12 gives rise to the cusp form $\Delta(q)$ of weight 12.

Most of the results of this paper were obtained by using computer. The author used Mathematica Ver. 3.0 for Linux.

1 Preliminaries

In this section, we recall or give some definitions and facts necessary in ths paper.

1.1 The Griess algebra of a vertex operator algebra

Let V be a vertex operator algebra (VOA) over the field $\mathbb{C}$ of complex numbers. Recall that it is a vector space equipped with a linear map

$$Y(a,z)=\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}\in(\operatorname{End}V)[[z,z^{-1}]]$$

and nonzero vectors $\mathbf{1}$ and ω satisfying a number of conditions [Bo1],[FLM2]. We recall some of the properties (cf. [MaN]). The operators $a_{(n)}$, $(a\in V, n\in\mathbb{Z})$, are subject to

$$\sum_{i=0}^{\infty}\binom{p}{i}(a_{(r+i)}b)_{(p+q-i)}=\sum_{i=0}^{\infty}(-1)^r\binom{r}{i}\Big(a_{(p+r-i)}b_{(q+i)}-(-1)^r b_{(q+r-i)}a_{(p+i)}\Big). \tag{1.1}$$

where $a,b,c\in V$ and $p,q,r\in\mathbb{Z}$. The *vacuum vector* $\mathbf{1}$ satisfies

$$\mathbf{1}_{(n)}a=\begin{cases}0, & (n\neq -1),\\ a, & (n=-1),\end{cases}\quad \text{and}\quad a_{(n)}\mathbf{1}=\begin{cases}0, & (n\geq 0),\\ a, & (n=-1).\end{cases} \tag{1.2}$$

The *conformal vector* ω generates a representation of the Virasoro algebra:

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}\,c$$

where $L_m=\omega_{(m+1)}$. The central charge c is called the *rank* of the VOA. The operator L_{-1} satisfies

$$(L_{-1}a)_{(n)}=(\omega_{(0)}a)_{(n)}=-na_{(n-1)}. \tag{1.3}$$

The operator L_0 is supposed to be semisimple giving rise to a grading $V=\bigoplus_{n=0}^{\infty}V^n$ of the VOA V, where V^n denotes the eigenspace with eigenvalue n, which is supposed to be finite-dimensional by definition. The eigenvalue is called the *degree*. It follows that $V^i_{(n)}V^j\subset V^{i+j-n-1}$ and $\mathbf{1}\in V^0$. We denote the sum of the subspaces with degree up to n as

$$V^{\leq n}=\bigoplus_{m=0}^{n}V^m. \tag{1.4}$$

Throughout the paper, we assume that the grading of the VOA V is of the form

$$V=\bigoplus_{n=0}^{\infty}V^n,\quad \text{where}\quad V^0=\mathbb{C}\mathbf{1}\quad\text{and}\quad V^1=0. \tag{1.5}$$

Consider the first nontrivial subspace $B=V^2$. Set

$$ab=a_{(1)}b,\quad (a|b)\mathbf{1}=a_{(3)}b,\quad \text{for } a,b\in B. \tag{1.6}$$

Then the multiplication gives a commutative nonassociative algebra structure on B and $(\ |\ )$ is a symmetric invariant bilinear form on it. The space B equipped with these structures is called the *Griess algebra* of V. We denote the adjoint action of an element $a\in B$ as

$$R_a\,:\,B\to B,\ x\mapsto xa\ (=ax). \tag{1.7}$$

By a slight abuse of terminology, we call a vector $e \in B$ satisfying $e_{(1)}e = 2e$ an *idempotent*. A vector $e \in B$ generates a representation of the Virasoro algebra on V if and only if it is an idempotent of B, for which the central charge is given by $2(e|e)$. The conformal vector ω of the VOA V is twice an identity element of the algebra, i.e., $\omega a = 2a$ for any $a \in B$. The squared norm $(\omega|\omega) = c/2$ is half the rank of the VOA V.

Recall that the VOA V carries a unique invariant bilinear form $(\ |\ )$ up to normalization ([Li]). We normalize the form by $(\mathbf{1}|\mathbf{1}) = 1$. It is indeed an extension of the form $(\ |\ )$ on B to the whole space V, so we have denoted it by the same symbol. We note that $(V^i|V^j) = 0$ if $i \neq j$. The relation

$$(a_{(n)}u|v) = (u|a_{(2m-2-n)}v), \quad (u, v \in V), \tag{1.8}$$

for a vector $a \in V^m$ such that $L_1 a = 0$ is a particular case of the invariance of the form. Therefore, $(a_{(n)}u|v) = (u|a_{(2-n)}v)$ holds for any $a \in B$ thanks to the assumption (1.5).

1.2 Virasoro submodule generated by the vacuum vector

Let V be a VOA and let V_ω be the Virasoro submodule generated by the vacuum vector $\mathbf{1}$ with respect to the action L_n associated with the conformal vector ω. Then, since $L_n\mathbf{1} = 0$ for $n \geq -1$, we have the following sequence of surjective homomorphisms of Virasoro modules:

$$M(c,0)/M(c,1) \to V_\omega \to L(c,0), \tag{1.9}$$

where $M(c,h)$ (resp. $L(c,h)$) denote the Verma module (resp. irreducible module) over the Virasoro algebra of central charge c with highest weight (lowest conformal weight) h. Let us denote the highest weight vector of $M(c,0)/M(c,1)$ mapped to $\mathbf{1} \in V_\omega \subset V$ by the same symbol $\mathbf{1}$.

Now, let P_n denote the set of all partitions of n by integers greater than 1:

$$P_n = \{\vec{m} = m_1 m_2 \cdots m_k \,|\, k, m_1, \ldots, m_k \in \mathbb{N},\ m_1 \geq m_2 \geq \cdots \geq m_k \geq 2\}. \tag{1.10}$$

For each partition $\vec{m} \in P_n$, we set

$$[\vec{m}] = [m_1, m_2, \ldots, m_k] = L_{-m_1} L_{-m_2} \cdots L_{-m_k}\mathbf{1} \in M(c,0)/M(c,1). \tag{1.11}$$

Then the set $\{[\vec{m}] \,|\, \vec{m} \in P_n\}$ forms a basis of the subspace with conformal weight n of the module $M(c,0)/M(c,1)$. We denote

$$L_{\vec{m}} = L_{m_k} \cdots L_{m_2} L_{m_1} \tag{1.12}$$

for a partition $\vec{m} = m_1 m_2 \cdots m_k$.

Recall that a singular vector (or a primary vector) of a Virasoro module is a nonzero vector v of the module such that

$$L_m v = 0 \quad \text{for all } m \geq 1. \tag{1.13}$$

In this paper, by convention, a *singular vector* means a nonzero vector in a highest weight module over the Virasoro algebra satisfying (1.13) which is not a multiple of the highest

weight vector (vacuum vector), and a *primary vector* means any nonzero vector in a VOA satisfying (1.13).

The module $M(c,0)/M(c,1)$ contains a singular vector of conformal weight up to n if and only if the central charge c is a zero of a certain reduced polynomial $D_n(c)$, which can be computed by the Kac-determinant formula. We normalize the polynomials for $n = 2, \dots, 10$ as follows.

$$\begin{aligned}
D_2(c) &= c,\\
D_4(c) &= c(5c+22),\\
D_6(c) &= c(2c-1)(5c+22)(7c+68),\\
D_8(c) &= c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68),\\
D_{10}(c) &= 10c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)(11c+232).
\end{aligned} \tag{1.14}$$

If $D_n(c) \neq 0$, then, up to the degree n, the maps (1.9) are isomorphisms and we may identify V_ω with $M(c,0)/M(c,1)$,

1.3 VOA of class $\mathcal{S}^n$

An automorphism of a VOA V is a linear isomorphism $g : V \to V$ satisfying $g(a_{(n)}b) = (ga)_{(n)}(gb)$ for all $a, b \in V$ and all $n \in \mathbb{Z}$ that fixes the conformal vector ω. Let $\operatorname{Aut} V$ denote the group of all automorphisms of V.

Any automorphism sends the vacuum vector $\mathbf{1}$ to itself and preserves the grading. Since an automorphism $g \in \operatorname{Aut} V$ fixes any vector in the subspace V_ω, the group $\operatorname{Aut} V$ acts on the quotient space V/V_ω and on its graded pieces V^n/V^n_ω.

Definition 1.1 A VOA V is said to be *of class* $\mathcal{S}^n$ if the action of $\operatorname{Aut} V$ on $V^{\le n}/V^{\le n}_\omega$ is fixed-point free.

In other words, a VOA V is of class $\mathcal{S}^n$ if $V^{\le n}$ has no extra fixed-vector other than those belonging to $V^{\le n}_\omega$.

The VOA $L(c,0)$ associated with the irreducible highest weight representation of the Virasoro algebra at central charge c (cf. [FrZh]) is obviously of class $\mathcal{S}^n$. However, since its Griess algebra B is one-dimensional, this example is not of our interest, although we will use $L(\frac{1}{2},0)$ etc. later in a different context.

The main example of our concern is the moonshine module $V^\natural$ constructed by Frenkel et al. in [FLM1] [FLM2]. It follows from [CN] and [Bo2] (cf. [HL] and [DM]) that the VOA $V^\natural$ is of class $\mathcal{S}^{11}$. We will later see that the moonshine module $V^\natural$, for which $c = 24$ and $\dim B = 196884$, has exceptionally large symmetry in our sense.

In the rest of this paper, we always assume that $D_n(c) \neq 0$ whenever V is supposed to be of class $\mathcal{S}^n$, for the cases when $D_n(c) = 0$ are not interesting from our point of view. The VOA's of our concern are at most of class $\mathcal{S}^{10}$ so that the excluded values of c are only 0, 1/2, −46/3, −3/5, −22/5, −68/7 and −232/11.

2 Trace Formulae for the Griess Algebra of a VOA

Let B be the Griess algebra of a VOA V of rank c, and let d denote the dimension of B.

In the first subsection, we give formulae that describes the traces $\operatorname{Tr} R_{a_1} R_{a_2} \cdots R_{a_m}$ up to $m = 5$ under appropriate assumptions. Sketch of the derivation of the formulae will be given in subsequent subsections.

2.1 The formulae

To describe the formulae, we set

$$(a_1|a_2|a_3) = (a_1|a_2 a_3), \tag{2.1}$$

which is a totally symmetric trilinear form on B, and define a totally antisymmetric quinery form on B by setting

$$(a_1, a_2, a_3, a_4, a_5)\mathbf{1} = \frac{1}{5!} \sum_{\sigma \in S_5} (-1)^{\ell(\sigma)} \sigma(a_{1(3)} a_{2(2)} a_{3(1)} a_{4(0)} a_5). \tag{2.2}$$

Here we let $\sigma \in S_5$ act by the permutation of the indices of a_i, $i = 1, \ldots, 5$.

Let Cyc denote the operation of summing over the cyclic permutations of the indices, and Sym denote that over all permutations such that the results are apparently distinct after performing the symmetries $a_i a_j = a_j a_i$, $(a_i|a_j) = (a_j|a_i)$ and $(a_i a_j|a_k) = (a_i|a_j a_k)$ for any $i, j, k = 1, \ldots, 5$; For instance,

$$\begin{aligned}
&\operatorname{Sym} (a_1|a_2)(a_3|\omega)(a_4|\omega) \\
&\quad = (a_1|a_2)(a_3|\omega)(a_4|\omega) + (a_1|a_3)(a_2|\omega)(a_4|\omega) + (a_1|a_4)(a_2|\omega)(a_3|\omega) \\
&\qquad + (a_2|a_3)(a_1|\omega)(a_4|\omega) + (a_2|a_4)(a_1|\omega)(a_3|\omega) + (a_3|a_4)(a_1|\omega)(a_2|\omega).
\end{aligned}$$

The result is summarized in the following theorem.

Theorem 2.1 *Let B be the Griess algebra of a VOA V such that the bilinear form $(\ |\ )$ on B is nondegenerate or $\operatorname{Aut} V$ is finite*[1]

(1) *If V is of class $\mathcal{S}^2$ then, for any $a \in B$,*

$$\operatorname{Tr} R_a = \frac{4d}{c}(a|\omega).$$

(2) *If V is of class $\mathcal{S}^4$ then, for any $a_1, a_2 \in B$,*

$$\operatorname{Tr} R_{a_1} R_{a_2} = \frac{-2(5c^2 - 88d + 2cd)}{c(5c+22)}(a_1|a_2) + \frac{4(5c+22d)}{c(5c+22)}(a_1|\omega)(a_2|\omega).$$

[1]We suppose this for simplicity although the condition can be slightly weakened.

(3) *If V is of class $\mathcal{S}^6$ then, for any $a_1, a_2, a_3 \in B$,*

$$\begin{aligned}
&\mathrm{Tr}\ R_{a_1}R_{a_2}R_{a_3} \\
&\quad= \frac{-3c^2(70c^2+769c-340)+2d(4c^3-445c^2+12236c-5984)}{c(2c-1)(5c+22)(7c+68)}(a_1|a_2|a_3) \\
&\qquad+\frac{4c(70c^2+1017c-340)-8d(32c^2-1419c+748)}{c(2c-1)(5c+22)(7c+68)}\,\mathrm{Cyc}\,(a_1|a_2)(a_3|\omega) \\
&\qquad\qquad+\frac{5952c(d-1)}{c(2c-1)(5c+22)(7c+68)}(a_1|\omega)(a_2|\omega)(a_3|\omega).
\end{aligned}$$

(4) *If V is of class $\mathcal{S}^8$ then, for any $a_1, a_2, a_3, a_4 \in B$,*

$$\begin{aligned}
&\mathrm{Tr}\ R_{a_1}R_{a_2}R_{a_3}R_{a_4} \\
&\quad= \frac{1}{D_8(c)}\Big(A_1(a_1a_2|a_3a_4)+A_2(a_1a_3|a_2a_4)+A_3(a_1a_4|a_3a_2) \\
&\qquad+B\,\mathrm{Sym}\,(a_1|a_2|a_3)(a_4|\omega)+C\,\mathrm{Sym}\,(a_1|a_2)(a_3|a_4) \\
&\quad+D\,\mathrm{Sym}\,(a_1|a_2)(a_3|\omega)(a_4|\omega)+E(a_1|\omega)(a_2|\omega)(a_3|\omega)(a_4|\omega)\Big)
\end{aligned}$$

where $D_8(c)$ is given in (1.14) *and the coefficients are listed in* Appendix A.1.

(5) *If V is of class $\mathcal{S}^{10}$ then, for any $a_1, a_2, a_3, a_4, a_5 \in B$,*

$$\begin{aligned}
&\mathrm{Tr}\ R_{a_1}R_{a_2}R_{a_3}R_{a_4}R_{a_5} \\
&\quad= \frac{1}{D_{10}(c)}\Big(\sum A_{i_1,i_2,i_3,i_4,i_5}(a_{i_1}a_{i_2}|a_{i_3}|a_{i_4}a_{i_3}) \\
&\quad+B_1\,\mathrm{Cyc}(a_1a_2|a_3a_4)(a_5|\omega)+B_2\,\mathrm{Cyc}(a_1a_3|a_2a_4)(a_5|\omega)+B_3\,\mathrm{Cyc}(a_1a_4|a_2a_3)(a_5|\omega) \\
&\quad+C\,\mathrm{Sym}(a_1|a_2|a_3)(a_4|a_5)+D\,\mathrm{Sym}(a_1|a_2|a_3)(a_4|\omega)(a_5|\omega) \\
&\quad+E\,\mathrm{Sym}(a_1|a_2)(a_3|a_4)(a_5|\omega)+F\,\mathrm{Sym}(a_1|a_2)(a_3|\omega)(a_4|\omega)(a_5|\omega) \\
&\quad+G(a_1|\omega)(a_2|\omega)(a_3|\omega)(a_4|\omega)(a_5|\omega)+H(a_1,a_2,a_3,a_4,a_5)\Big)
\end{aligned}$$

where the summation is over the permutations of $(1,2,3,4,5)$ for which $(a_{i_1}a_{i_2}|a_{i_3}|a_{i_4}a_{i_5})$ are distinct. The determinant $D_{10}(c)$ is given in (1.14) *and the coefficients are listed in* Appendix A.2.

Remark 2.2 Suppose that V is of class $\mathcal{S}^8$. By the cyclic property of trace:

$$\mathrm{Tr}\,R_{a_1}R_{a_2}R_{a_3}R_{a_4} = \mathrm{Tr}\,R_{a_2}R_{a_3}R_{a_4}R_{a_1},$$

we must have $A_1 = A_3$ in Theorem 1 (4) if there exist elements $a_1, \dots, a_4$ such that $(a_1a_2|a_3a_4) \neq (a_1a_4|a_2a_3)$. In this case, the dimension is determined from the rank as

$$d = -\frac{1050\ c^6+22565\ c^5+33121\ c^4-1707790\ c^3-3390408\ c^2+308160\ c}{2\ (30\ c^5-3212\ c^4+107355\ c^3-1135590\ c^2-206024\ c+825792)}. \tag{2.3}$$

We will discuss restrictions on the pair (c,d) similar to (2.3) later in Section 3.

2.2 Derivation by Casimir elements

In this subsection, we sketch the derivation of the formulae in case the form $(\,|\,)$ on B is nondegenerate.

Let $\{x_1, \dots, x_d\}$ be a basis of B and let $\{x^1, \dots, x^d\}$ be the dual basis with respect to the form $(\,|\,)$: $(x_i|x^j) = \delta_{i,j}$. We suppose that any expression with a repeated index i must be summed over $i = 1, \dots, d$. Note that

$$(x^i_{(3)}a)_{(-1)}x_i = x^i_{(-1)}a_{(3)}x_i = a. \tag{2.4}$$

The strategy of the derivation of the formula is to write the trace as

$$\mathrm{Tr}\, R_{a_1} \cdots R_{a_m} = (a_{1(1)} \cdots a_{m(1)}x_i|x^i) = (x^i_{(1)}a_{m-1(1)} \cdots a_{1(1)}x_i|a_m),$$

and to rearrange the vectors by using the identity (1.1) and the invariance of the form $(\,|\,)$ on V until the trace is written as a sum of expressions of the form $(x^i_{(k)}x_i|X)$ where X is an element of V written by $a_1, a_2, \dots, a_m$ and k is an integer.

Let us first study the 'Casimir' elements:

$$\kappa_n = x^i_{(3-n)}x_i = \sum_{i=1}^{d} x^i_{(3-n)}x_i, \tag{2.5}$$

which do not depend on the choice of the basis. Note that

$$\kappa_0 = d\mathbf{1} \quad \text{and} \quad \kappa_1 = 0. \tag{2.6}$$

and that the vector κ_n for an odd n is determined from those for even n by the action of L_{-1}. By the identity (1.1), the sequence $\kappa_0, \dots, \kappa_n$ is subject to the relations

$$L_m\kappa_k = (m+k-2)\kappa_{k-m} + \delta_{m,2}L_{-k+2}\mathbf{1} + \delta_{m,k-2}\frac{m^3-m}{6}L_{-2}\mathbf{1}, \ (k = 0, \dots, n). \tag{2.7}$$

Now, in order to deduce some information on the Griess algebra by these elements, the following simple observation is fundamental.

Lemma 2.3 *The vector $\kappa_n = x^i_{(3-n)}x_i$ is fixed by any automorphism of the VOA V.*

Therefore, if the VOA V has larger symmetry then the vector κ_n has to belong to a smaller subspace of V; the smallest possible case is V_ω when the VOA is of class $\mathcal{S}^n$.

Lemma 2.4 *If the VOA V is of class $\mathcal{S}^n$ then the Casimir elements $\kappa_2, \dots, \kappa_n$ are contained in V_ω.*

Now, suppose that the vectors $\kappa_2, \dots, \kappa_n$ are indeed contained in V_ω. If the Virasoro submodule V_ω does not contain a singular vector of degree up to n, then these vectors are uniquely determined by the properties (2.6) and (2.7). In this way, we have the following result.

Proposition 2.5 *If the VOA V is of class $\mathcal{S}^n$ then κ_n is uniquely written as*

$$\kappa_n = \frac{1}{D_n(c)} \sum_{\vec{m} \in P_n} P_{\vec{m}}(c,d)[\vec{m}]. \tag{2.8}$$

where $P_{\vec{m}}(c,d)$ are certain polynomials in c and d.

The explicit expressions of κ_n for $n = 2, 4, 6$ are given as follows:

$$\begin{aligned}
\kappa_2 &= \frac{4d}{c}[2], \quad \kappa_4 = \frac{6(d-1)}{5c+22}[4] + \frac{2(5c+22d)}{c(5c+22)}[2,2], \\
\kappa_6 &= \frac{8(d-1)(5c^2+35c-228)}{(2c-1)(5c+22)(7c+68)}[6] \\
&\quad + \frac{2c(70c^2+769c+1644)+4d(92c^2+427c-748)}{c(2c-1)(5c+22)(7c+68)}[4,2] \\
&\quad + \frac{31(d-1)(5c+44)}{(2c-1)(5c+22)(7c+68)}[3,3] + \frac{992(d-1)}{(2c-1)(5c+22)(7c+68)}[2,2,2].
\end{aligned} \tag{2.9}$$

The expressions for higher n are so lengthy; we do not include them in this paper. The expressions in case $c = 24$ and $d = 196884$ look as follows:

$$\begin{aligned}
\kappa_2 &= 32814[2], \quad \kappa_4 = 8319[4] + 2542[2,2], \\
\kappa_6 &= 3492[6] + 1302[4,2] + \frac{1271}{2}[3,3] + 124[2,2,2], \\
\kappa_8 &= \frac{3863}{2}[8] + 552[6,2] + 434[5,3] + \frac{333}{2}[4,4] + 96[4,2,2] + 93[3,3,2] + \frac{13}{3}[2,2,2,2], \\
\kappa_{10} &= 1182[10] + \frac{613}{2}[8,2] + 207[7,3] + 141[6,4] + 41[6,2,2] + 74[5,5] + 64[5,3,2] \\
&\quad + \frac{99}{4}[4,4,2] + 24[4,3,3] + \frac{9}{2}[4,2,2,2] + \frac{13}{2}[3,3,2,2] + \frac{7}{60}[2,2,2,2,2].
\end{aligned}$$

Now, suppose that V is of class $\mathcal{S}^2$ and let a be any element of the Griess algebra. Then we immediately obtain Theorem 1 (1): $\operatorname{Tr} R_a = (a_{(1)}x^i|x_i) = (x^i_{(1)}x_i|a) = 4d(a|\omega)/c$. In particular, we have

$$\operatorname{Tr} R_{ab} = \frac{8d}{c}(a|b). \tag{2.10}$$

Next, take any two elements a, b of the Griess algebra. Then

$$\operatorname{Tr} R_a R_b = (a_{(1)}b_{(1)}x^i|x_i) = (x^i_{(1)}a_{(1)}x_i|b).$$

By the identity (1.1) for $p = -1, q = 1, r = 2$, we have $-(x^i_{(3)}a)_{(-1)}x_i = x^i_{(1)}a_{(1)}x_i + x^i_{(-1)}a_{(3)}x_i - a_{(3)}x^i_{(-1)}x_i + 2a_{(2)}x^i_{(0)}x_i - a_{(1)}x^i_{(1)}x_i$. Since $a_{(2)}x^i_{(0)}x_i = a_{(1)}x^i_{(1)}x_i$,

$$\begin{aligned}
\operatorname{Tr} R_a R_b &= -2(a|b) + (a_{(3)}x^i_{(-1)}x_i|b) - (a_{(1)}x^i_{(1)}x_i|b) \\
&= -2(a|b) + (x^i_{(-1)}x_i|a_{(-1)}b) - (x^i_{(1)}x_i|a_{(1)}b)
\end{aligned}$$

Suppose that V is of class $\mathcal{S}^4$ and substitute the expressions of $\kappa_2 = x^i_{(1)}x_i$ and $\kappa_4 = x^i_{(-1)}x_i$ given by (2.9). Then using

$$\begin{aligned}(L_{-4}\mathbf{1}|a_{(-1)}b) &= (\mathbf{1}|L_4 a_{(-1)}b) = 6(a|b),\\ (L_{-2}L_{-2}\mathbf{1}|b_{(-1)}a) &= (\mathbf{1}|L_2L_2(b_{(-1)}a)) = 2(a|\omega)(b|\omega) + 8(a|b),\end{aligned} \tag{2.11}$$

we have Theorem 1 (2):

$$\begin{aligned}&\operatorname{Tr} R_a R_b\\ &\quad= -2(a|b) + \frac{6(d-1)}{5c+22}6(a|b) + \frac{2(5c+22d)}{c(5c+22)}\Big(2(a|\omega)(b|\omega) + 8(a|b)\Big) - \frac{4d}{c}(ab|\omega)\\ &\quad= \frac{-2(5c^2-88d+2cd)}{c(5c+22)}(a|b) + \frac{4(5c+22d)}{c(5c+22)}(a|\omega)(b|\omega)\end{aligned}$$

Similarly, one can obtain a expression in terms of the inner product and the multiplication of the trace in which three and four elements of the Griess algebra are concerned if V is of class $\mathcal{S}^6$ and of $\mathcal{S}^8$, respectively. However, if five elements are concerned, then we encounter the expression $a_{1(3)}a_{2(2)}a_{3(1)}a_{4(0)}a_5$ and its permutations which cannot be written by a combination of the inner product $(\ |\ )$ and the multiplication in general. Thus we are led to consider the totally antisymmetric quinery form defined by (2.2). Then the trace $\operatorname{Tr} R_{a_1}R_{a_2}R_{a_3}R_{a_4}R_{a_5}$ is written by a combination of these operations if V is of class $\mathcal{S}^{10}$. In this way, we obtain Theorem 1 (3)–(5).

2.3 Derivation by projection to V_ω

In this subsection, we will sketch another derivation of the formulae under the assumption that $\operatorname{Aut} V$ is finite.

For any $v \in V$, let $\tilde{v}$ denote its average over the action of the automorphism group:

$$\tilde{v} = \frac{1}{|\operatorname{Aut} V|}\sum_{g\in\operatorname{Aut} V} gv. \tag{2.12}$$

The following lemma is obvious.

Lemma 2.6 *Let v be an element of V^n. Then $\operatorname{Tr}|_{V^k} v_{(n-1)} = \operatorname{Tr}|_{V^k}(gv)_{(n-1)}$ for any $g \in \operatorname{Aut} V$ at any degree k.*

In particular, we have $\operatorname{Tr} v_{(n-1)} = \operatorname{Tr}\tilde{v}_{(n-1)}$ for any $v \in V^n$.

Now, for each n, consider the map

$$\eta_n : V^n \to \mathbb{C}^{\dim V^n}, \quad v \mapsto (L_{\vec{m}}v)_{\vec{m}\in P_n}, \tag{2.13}$$

where P_n is the set (1.10) which parametrizes a basis of V^n_ω.

Lemma 2.7 *If $D_n(c) \neq 0$ then $V^n = V^n_\omega \oplus \operatorname{Ker}\eta_n$.*

Proof. Since the map η_n is isomorphic on V^n_ω if $D_n(c) \neq 0$, the result follows. *QED.*

For any $v \in V$, let $\delta(v)$ denote its projection to V_ω with respect to the decomposition as in the lemma.

Lemma 2.8 *Suppose that V is of class $\mathcal{S}^n$. Then $\mathrm{Tr}\,|_{V^k} v_{(n-1)} = \mathrm{Tr}\,|_{V^k} \delta(v)_{(n-1)}$ for any $v \in V^n$ at any degree k.*

Proof. Set $\delta = \delta(v)$ and $\pi = v - \delta(v)$. Then obviously $\tilde{v} = \tilde{\delta} + \tilde{\pi} = \delta + \tilde{\pi}$, and $\tilde{\pi}$ is fixed by any automorphism of V. Therefore, if V is of class $\mathcal{S}^n$ then $\tilde{\pi} = 0$ because $\tilde{\pi} \in V_\omega \cap \mathrm{Ker}\,\eta_n$. Hence, by Lemma 2.6, we have $\mathrm{Tr}\,|_{V^k} v_{(n-1)} = \mathrm{Tr}\,|_{V^k} \tilde{v}_{(n-1)} = \mathrm{Tr}\,|_{V^k} \delta_{(n-1)}$. *QED.*

Since $\delta(v) \in V_\omega$, the action $\delta(v)_{(n-1)}$ on B can be explicitly computed by the commutation relation of the Virasoro algebra.

For example, if $c \neq 0$, then we have $\delta(a) = 2(a|\omega)\omega/c$ for any $a \in B$. Namely, any element $a \in B$ can be written as

$$a = \frac{2(a|\omega)}{c}\omega + \pi, \tag{2.14}$$

where π is a primary vector in $B = V^2$. If V is of class $\mathcal{S}^2$ then

$$\mathrm{Tr}\, R_a = \frac{2(a|\omega)}{c} \mathrm{Tr}\, R_\omega = \frac{4d}{c}(a|\omega).$$

Thus we have obtained Theorem 1 (1).

Next, let a, b be any elements of B. Then, by the identity (1.1) for $p = 2$, $q = 1$, $r = -1$, we have $a_{(1)}b_{(1)} = (a_{(-1)}b)_{(3)} + 2(a_{(0)}b)_{(2)} + (a_{(1)}b)_{(1)} - a_{(-1)}b_{(3)} - b_{(-1)}a_{(3)}$ on B. Therefore,

$$\mathrm{Tr}\, R_a R_b = \mathrm{Tr}\,|_B (a_{(-1)}b)_{(3)} + 2\mathrm{Tr}\,|_B (a_{(0)}b)_{(2)} + \mathrm{Tr}\,|_B (a_{(1)}b)_{(1)} - 2(a|b).$$

If V is of class $\mathcal{S}^4$ then, since

$$\begin{aligned} \delta(a_{(-1)}b) &= \frac{6c(a|b) - 12(a|\omega)(b|\omega)}{c(5c+22)}[4] + \frac{44(a|b) + 20(a|\omega)(b|\omega)}{c(5c+22)}[2,2], \\ \delta(a_{(0)}b) &= \frac{2(a|b)}{c}[3], \quad \delta(a_{(1)}b) = \frac{4(a|b)}{c}[2] \end{aligned} \tag{2.15}$$

and $\mathrm{Tr}\,|_B[4]_{(3)} = 6d$, $\mathrm{Tr}\,|_B[2,2]_{(3)} = 8d + c$, $\mathrm{Tr}\,|_B[3]_{(2)} = -4d$, $\mathrm{Tr}\,|_B[2]_{(1)} = 2d$, we have Theorem 1 (2).

The derivation of Theorem 1 (3)–(5) is similar.

3 Constraints on c and d

Let B be the Griess algebra of a VOA V such that the form $(\ |\ )$ on B is nondegenerate.

In this section, we will give some necessary conditions satisfied by the pair (c, d) for a VOA with larger symmetry under some additional conditions on V.

3.1 Constraints from a proper idempotent

By a *proper idempotent* we mean an idempotent $e \in B$ such that the central charge $b = 2(e|e)$ differs from 0 and c. If the algebra B has a real form on which the bilinear form $(\ |\ )$ is positive-definite then, by Theorem 11 of [MeN] and Theorem 6.8 of [Mi1], the conformal vector ω is decomposable if $d \geq 2$; In particular, the algebra B contains a proper idempotent.

Lemma 3.1 *Let $\varphi(e)$ be a vector generated by an idempotent $e \in B$. Then*

$$(e_{(2)}(a_{(m-1)}b)|\varphi(e)) = (3-m)(a_{(m)}b|\varphi(e))$$

holds for any $a, b \in B$ and $m \in \mathbb{Z}$.

Proof. Note that the actions of $e_{(p)}$ and $(\omega - e)_{(q)}$ commute, and $(\omega - e)_{(q)}\mathbf{1} = 0$ if $q \geq 0$. Hence $e_{(q)}\varphi(e) = \omega_{(q)}\varphi(e)$ if $q \geq 0$. Therefore, by the invariance of the bilinear form $(\ |\ )$ on V, $(e_{(2)}(a_{(n)}b)|\varphi(e)) = (a_{(n)}b|e_{(0)}\varphi(e)) = (a_{(n)}b|\omega_{(0)}\varphi(e)) = (\omega_{(2)}(a_{(n)}b)|\varphi(e))$. The result follows from $(\omega_{(2)}(a_{(m-1)}b)|\varphi(e)) = ((\omega_{(0)}a)_{(m+1)}b)|\varphi(e)) + 2((\omega_{(1)}a)_{(m+1)}b)|\varphi(e)) = (3-m)(a_{(m)}b)|\varphi(e))$. *QED.*

Now suppose that V is of class $\mathcal{S}^6$. By the lemma,

$$(x^i_{(-1)}e_{(1)}x_i|e_{(0)}e_{(0)}e) = 3(x^i_{(0)}e_{(1)}x_i|e_{(0)}e). \tag{3.1}$$

Computing the both-hand sides of this equality by the method of subsection 2.2, we have

$$b(b-c)\Big((70c^2 + 955c + 2388)c - 2(c^2 - 55c + 748)d\Big) = 0,$$

where $b = 2(e|e)$ is the central charge of e. Therefore, if e is proper then

$$d = \frac{(70c^2 + 955c + 2388)c}{2(c^2 - 55c + 748)} = 35c + 2402 + \frac{c^2 + 214248c - 3593392}{2(c^2 - 55c + 748)}. \tag{3.2}$$

Further, if V is of class $\mathcal{S}^8$ then, by computing

$$(x^i_{(-3)}e_{(1)}x_i|e_{(0)}e_{(0)}e_{(0)}e_{(0)}e) = 5(x^i_{(-2)}e_{(1)}x_i|e_{(0)}e_{(0)}e_{(0)}e), \tag{3.3}$$

we have

$$d = \frac{5250c^5 + 155250c^4 + 1369715c^3 + 3507098c^2 + 1497768c}{125c^4 - 4770c^3 - 23382c^2 + 1561868c + 1032240} \tag{3.4}$$

if e is proper.

Theorem 3.2 *Let V be a VOA of class $\mathcal{S}^8$ for which the form $(\ |\ )$ on B is nondegenerate. If B contains a proper idempotent then $c = 24$ and $d = 196884$.*

Proof. By (3.2) and (3.4), $c = -46/3,\ -68/7,\ -22/5,\ -3/5,\ 0,\ 1/2,\ 24,\ 142/5$. However, $D_6(c) = 0$ for the first 6 cases and $d < 0$ for the last case. *QED.*

By the remark at the beginning of this subsection, we have the following corollary.

Corollary 3.3 *Suppose that the algebra B has a real form on which the bilinear form $(\ |\ )$ is positive-definite. If the VOA V is of class $\mathcal{S}^8$ and if $d \geq 2$ then $c = 24$ and $d = 196884$.*

Now, let us come back to VOA's of class $\mathcal{S}^6$ but restrict our attention to the case when the rank c is a positive half-integer. In this case, inspecting the relation (3.2), we see that the pair (c, d) must be one from the following table:

c	d	c	d	c	d	c	d
8	156	$23\frac{1}{2}$	96256	32	139504	$54\frac{1}{2}$	9919
16	2296	24	196884	34	57889	68	8146
20	10310	$24\frac{1}{2}$	1107449	36	35856	$93\frac{1}{2}$	7566
$21\frac{1}{2}$	21414	$30\frac{1}{2}$	1964871	40	20620	132	8154
22	28639	$31\frac{1}{2}$	207144	44	14994	1496	54836

Table 3.1

3.2 Constraints from an idempotent of central charge 1/2

Suppose that B contains an idempotent of central charge $1/2$ for which the eigenvalues of the adjoint action are $0, 1/2, 1/16$ and 2, and the eigenspace with eigenvalue 2 is one-dimensional. This is indeed the case if e generates a subVOA isomorphic to $L(\frac{1}{2}, 0)$, for this is a rational VOA for which the irreducible modules are isomorphic to either $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, \frac{1}{2})$ or $L(\frac{1}{2}, \frac{1}{16})$. In particular, this holds if V has a real form on which the form $(\ |\ )$ is positive-definite and e is a real idempotent of central charge $1/2$.

First consider the case when the eigenspace with eigenvalue $1/16$ is zero. In this case, we have the following equations satisfied by the dimension $d(\frac{1}{2})$ of the eigenspace:

$$d(\tfrac{1}{2}) + 4 = 2\mathrm{Tr}\, R_e,\ \ d(\tfrac{1}{2}) + 16 = 4\mathrm{Tr}\, R_e^2 \tag{3.5}$$

By the compatibility, we obtain $2\mathrm{Tr}\, R_e^2 - \mathrm{Tr}\, R_e = 6$. If the VOA V is of class $\mathcal{S}^4$ then, by Theorem 1,

$$(-22 + 2c)d = (-37 - 10c). \tag{3.6}$$

Further if V is of class $\mathcal{S}^6$ then we get

$$(3c^2 + 164c - 2992)d = c(-140c^2 - 1903c - 4832). \tag{3.7}$$

By solving these equations, we obtain the following result.

Proposition 3.4 *Suppose that the VOA V contains an idempotent of central charge $1/2$ for which the eigenvalues are $0, 1/2$ and 2 and the eigenspace with eigenvalue 2 is one-dimensional. If V is of class $\mathcal{S}^6$ and $d \geq 2$ then $c = 8$ and $d = 156$.*

Now suppose that the rank c is a positive half-integer. If V is of class $\mathcal{S}^4$ then, by (3.6), the nonnegativity and the integrality of $d(0)$ and $d(\frac{1}{2})$ gives us a list of possible pairs of such c and d. They are given by the following table:

c	$d =$	$d(0) +$	$d(\frac{1}{2})$	$+\ 1$
$0\,\frac{1}{2}$	$1 =$	$0 +$	0	$+\ 1$
4	$22 =$	$14 +$	7	$+\ 1$
$7\,\frac{1}{2}$	$120 =$	$91 +$	28	$+\ 1$
8	$156 =$	$120 +$	35	$+\ 1$
$9\,\frac{1}{2}$	$418 =$	$333 +$	84	$+\ 1$
10	$685 =$	$551 +$	133	$+\ 1$
$10\,\frac{1}{2}$	$1491 =$	$1210 +$	280	$+\ 1$

Table 3.2

Note 3.5 The fixed-point VOA $V^+_{\sqrt{2}E_8}$ of the VOA associated with the lattice $\sqrt{2}E_8$ by the -1 automorphism of the lattice would be an example with $c = 8$ and $d = 156$. According to [Gr2], the automorphism group of this VOA is isomorphic to $O^+_{10}(2)$. The decomposition $156 = 120 + 35 + 1$ coincides with Theorem 5.2 of [Gr2]. The Hamming code VOA V_{H_8} considered by Miyamoto [Mi2], isomorphic to the VOA $V^+_{\sqrt{2}D_4}$, would be an example with $c = 4$ and $d = 22$. The automorphism group of this VOA is isomorphic to a group of shape $2^6 : (GL_3(2) \times S_3)$ [MaM].

We next consider the case when the 1/16 components are indeed present. In this case, if V is of class $\mathcal{S}^6$ then we have

$$(2c^2 - 110c + 1496)d = (70c^2 + 955c + 2388)c, \tag{3.8}$$

which is actually the same as the condition (3.2).

If V is of class $\mathcal{S}^6$ with $d \geq 2$ and if c is a positive half-integer then the rank c and the dimension d must be a pair from the following table:

c	$d =$	$d(0) +$	$d(\frac{1}{2}) +$	$d(\frac{1}{16})$	$+\ 1$
16	$2296 =$	$1116 +$	$155 +$	1024	$+\ 1$
20	$10310 =$	$4914 +$	$403 +$	4992	$+\ 1$
$23\,\frac{1}{2}$	$96256 =$	$46851 +$	$2300 +$	47104	$+\ 1$
24	$196884 =$	$96256 +$	$4371 +$	96256	$+\ 1$
$24\,\frac{1}{2}$	$1107449 =$	$543960 +$	$22816 +$	540672	$+\ 1$
$30\,\frac{1}{2}$	$1964871 =$	$1029630 +$	$13640 +$	921600	$+\ 1$
$31\,\frac{1}{2}$	$207144 =$	$109771 +$	$1116 +$	96256	$+\ 1$
32	$139504 =$	$74340 +$	$651 +$	64512	$+\ 1$
36	$35856 =$	$19951 +$	$0 +$	15904	$+\ 1$

Table 3.3

Note 3.6 The moonshine module $V^\natural$ is of course an example with $c = 24$ and $d = 196884$. The bosonic projection of the Babymonster VOSA $VB^\natural$ constructed by Höhn [Hö] would be an example with $c = 23\frac{1}{2}$ and $d = 96256$. The fixed-point VOA $V_{\Lambda_{16}}^{+}$ of the VOA associated with the Barnes-Wall lattice Λ_{16} by the -1 automorphism of the lattice would be an example with $c = 16$ and $d = 2296$.

4 Application to the Moonshine Module

Now, let $V^\natural$ be the moonshine module and let $B^\natural$ be the Conway-Griess algebra. In this section, we will compute the spectrum of the eigenspace decomposition of $B^\natural$ with respect to idempotents related to some Monster elements starting from the trace formulae by the representation theory of various subVOA's inside $V^\natural$.

4.1 Norton's formula

Since the moonshine module $V^\natural$ is of class $\mathcal{S}^{11}$, just substituting $c = 24$ and $d = 196884$ in Theorem 1, we recover the original trace formulae of Norton [No]:

Corollary 4.1 *For any elements a_1, a_2, a_3, a_4, a_5 of the Conway-Griess algebra $B^\natural$,*

$$\begin{aligned}
\operatorname{Tr} R_{a_1} &= 32814(a_1|\omega),\\
\operatorname{Tr} R_{a_1}R_{a_2} &= 4620(a_1|a_2) + 5084(a_1|\omega)(a_2|\omega),\\
\operatorname{Tr} R_{a_1}R_{a_2}R_{a_3} &= 900(a_1|a_2|a_3) + 620\,\mathrm{Cyc}\,(a_1|a_2)(a_3|\omega) + 744(a_1|\omega)(a_2|\omega)(a_3|\omega),\\
\operatorname{Tr} R_{a_1}R_{a_2}R_{a_3}R_{a_4} &= 166(a_1a_2|a_3a_4) - 116(a_1a_3|a_2a_4) + 166(a_1a_4|a_2a_3)\\
&\quad +114\,\mathrm{Sym}\,(a_1|a_2|a_3)(a_4|\omega) + 52\,\mathrm{Sym}\,(a_1|a_2)(a_3|a_4)\\
&\quad +80\,\mathrm{Sym}\,(a_1|a_2)(a_3|\omega)(a_4|\omega) + 104(a_1|\omega)(a_2|\omega)(a_3|\omega)(a_4|\omega),\\
\operatorname{Tr} R_{a_1}R_{a_2}R_{a_3}R_{a_4}R_{a_5} &= 30\mathrm{Cyc}\,(a_1a_2|a_3|a_4a_5) + 4\mathrm{Cyc}\,(a_1a_4|a_3|a_2a_5) - 22\mathrm{Cyc}\,(a_1a_5|a_3|a_2a_4)\\
&\quad +20\mathrm{Cyc}\,(a_1a_2|a_3a_4)(a_5|\omega) - 14\mathrm{Cyc}\,(a_1a_3|a_2a_4)(a_5|\omega) + 20\mathrm{Cyc}\,(a_1a_4|a_2a_3)(a_5|\omega)\\
&\quad + 8\,\mathrm{Sym}\,(a_1|a_2|a_3)(a_4|a_5) + 14\,\mathrm{Sym}\,(a_1|a_2|a_3)(a_4|\omega)(a_5|\omega)\\
&\quad + 6\,\mathrm{Sym}\,(a_1|a_2)(a_3|a_4)(a_5|\omega) + 10\,\mathrm{Sym}\,(a_1|a_2)(a_3|\omega)(a_4|\omega)(a_5|\omega)\\
&\quad +14(a_1|\omega)(a_2|\omega)(a_3|\omega)(a_4|\omega)(a_5|\omega) + 52(a_1, a_2, a_3, a_4, a_5).
\end{aligned}$$

Note 4.2 To compare this result with Table 2 of [No], substitute $1 = \omega/2$ (the identity element of the algebra) and suppose for the last one that a_1, a_2, a_3, a_4, a_5 are perpendicular to ω. The formula for $\operatorname{Tr} R_{a_1}R_{a_2}$ was given in p. 528 of [Co].

In particular, letting $a_1 = \cdots = a_5$ be an idempotent e, we have

$$\begin{aligned}
\operatorname{Tr}\ R_e &= 32814\,(e|e)\,,\\
\operatorname{Tr} R_e^2 &= 4620\,(e|e) + 5084\,(e|e)^2\,,\\
\operatorname{Tr} R_e^3 &= 1800\,(e|e) + 1860\,(e|e)^2 + 744\,(e|e)^3,\\
\operatorname{Tr} R_e^4 &= 864\,(e|e) + 1068\,(e|e)^2 + 480\,(e|e)^3 + 104\,(e|e)^4\,,\\
\operatorname{Tr} R_e^5 &= 480\,(e|e) + 680\,(e|e)^2 + 370\,(e|e)^3 + 100\,(e|e)^4 + 14\,(e|e)^5.
\end{aligned} \tag{4.1}$$

Recall that $2\,(e|e)$ is the central charge of the Virasoro algebra corresponding to e.

Remark 4.3 By a slight variation of the argument of Subsection 2.3, using results of Zhu [Zh] and Dong and Mason [DM] and the absence of a cusp form of weight less than 12, we see that Norton's formulae hold for any rational selfdual (holomorphic) VOA of rank 24 with shape (1.5) satisfying the C_2 finiteness condition.

4.2 Eigenspace decomposition of the Conway-Griess algebra

Let $U \to V^\natural$ be an inclusion of a VOA into $V^\natural$ such that

(1) The map preserves the operations of VOA's.

(2) The vacuum vector of U is mapped to the vacuum vector of $V^\natural$.

(3) The conformal vector t of U is mapped to an idempotent e in $B^\natural$.

(4) The action $e_{(n)}$ for nonnegative n coincides with that of $\omega_{(n)}$ on the image of U.

Recall that $V^\natural$ has a real form $V^\natural_{\mathbb{R}}$ on which the form $(\ |\ )$ is positive-definite. We suppose that e above is a real idempotent. Then the adjoint action $R_e = e_{(1)}$ is semisimple on $B^\natural$. Let $B^\natural(\lambda)$ denote the eigenspace of R_e with eigenvalue λ, and let $d(\lambda)$ be its dimension.

For the representation theory of various VOA's discussed below, we refer the reader to [FrZh], [Wa], [DMZ], [KMY], [Mi4], [DN], [Ab] as well as the physics papers [BPZ], [FaZ], [ZaF], [FaL], and for the construction of an automorphism by means of fusion rules to [Mi1] and [Mi4]. We will use the ATLAS notation [Atlas] for conjugacy classes of the Monster.

$L(\frac{1}{2}, 0)$ and 2A involution

Let U be the VOA $L(\frac{1}{2}, 0)$ associated with the irreducible highest weight representation of the Virasoro algebra at $c = 1/2$. There are 3 irreducible modules for this VOA, which are parametrized by the lowest conformal weight $h = 0,\ 1/16,\ 1/2$. Hence we have a decomposition

$$B^\natural = B^\natural(0) \oplus B^\natural(\tfrac{1}{16}) \oplus B^\natural(\tfrac{1}{2}) \oplus B^\natural(2),$$

where $d(2) = 1$. Hence

$$\frac{d(\frac{1}{2})}{2} + \frac{d(\frac{1}{16})}{16} + 2 = \mathrm{Tr}\, R_e, \quad \frac{d(\frac{1}{2})}{2^2} + \frac{d(\frac{1}{16})}{16^2} + 2^2 = \mathrm{Tr}\, R_e^2.$$

By (4.1) with $2(e|e) = 1/2$, we get

$$d(0) = d(\tfrac{1}{16}) = 96256,\ d(\tfrac{1}{2}) = 4371,\ d(2) = 1.$$

Now consider the map

$$1 \quad \text{on} \quad B^\natural(0) \oplus B^\natural(\tfrac{1}{2}) \oplus B^\natural(2), \quad -1 \quad \text{on} \quad B^\natural(\tfrac{1}{16}).$$

This map gives rise to an automorphism of $B^\natural$ [Mi1], so an involution of the Monster. It is identified with a 2A involution of the Monster by [Mi1] and [Co]. We may confirm this

by looking at the trace of this map; it is $96256-96256+4371+1=4372$, which coincides with the corresponding value in the list of Conway and Norton [CN].

Thus we have come back to the situation considered in Section 15 of [Co] without using any explicit construction of the Conway-Griess algebra.

$L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$ and 2B involution

Suppose given an embedding of $U=L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$ into $V^\natural$, and let e^1 and e^2 denote the images of the conformal vector of the first and the second component which we suppose to be real. Since they are mutually orthogonal, we have a simultaneous eigenspace decomposition

$$B^\natural=\bigoplus_{h,h'\in\{0,\frac{1}{16},\frac{1}{2},2\}}B^\natural(h,h').$$

We already know that $d(2,0)=d(0,2)=1$ and $d(2,\frac{1}{16})=d(2,\frac{1}{2})=d(\frac{1}{16},2)=d(\frac{1}{2},2)=d(2,2)=0$. By Corollary 4.1, we have

$$\operatorname{Tr}R_{e^1}R_{e^2}=\frac{1271}{4},\ \operatorname{Tr}R_{e^1}^2R_{e^2}=\operatorname{Tr}R_{e^1}R_{e^2}^2=\frac{403}{8},\ \operatorname{Tr}R_{e^1}^2R_{e^2}^2=\frac{197}{32}.$$

Therefore, the dimensions of the eigenspaces are given by

$$\begin{aligned}&d(0,0)=46851,\ d(0,\tfrac{1}{16})=d(\tfrac{1}{16},0)=d(\tfrac{1}{16},\tfrac{1}{16})=47104,\\&d(\tfrac{1}{2},0)=d(0,\tfrac{1}{2})=2300,\ d(\tfrac{1}{2},\tfrac{1}{16})=d(\tfrac{1}{2},\tfrac{1}{16})=2048,\ d(\tfrac{1}{2},\tfrac{1}{2})=23.\end{aligned}$$

Therefore, for the idempotent $e=e^1+e^2$,

$$\begin{aligned}&d(0)=46851,\ d(\tfrac{1}{16})=94208,\ d(\tfrac{1}{8})=47104,\ d(\tfrac{1}{2})=4600,\\&d(\tfrac{9}{16})=4096,\ d(1)=23,\ d(2)=2.\end{aligned}$$

In particular, the trace of the map

$$\begin{aligned}1\quad&\text{on}\quad B^\natural(0)\oplus B^\natural(\tfrac{1}{8})\oplus B^\natural(\tfrac{1}{2})\oplus B^\natural(1)\oplus B^\natural(2),\\-1\quad&\text{on}\quad B^\natural(\tfrac{1}{16})\oplus B^\natural(\tfrac{9}{16}),\end{aligned}$$

which is the composition of two 2A involutions corresponding to e^1 and e^2, is equal to $46851-94208+47104+4600-4096+23+2=276$. Hence this map is identified with a 2B involution of the Monster.

We may do the same analysis for an embedding of $L(\frac{1}{2},0)^{\otimes 3}$. However, the spectrum is not uniquely determined by the trace formulae; there are two possibilities.

$L(\frac{7}{10},0)$ and 2A involution

Let U be the VOA $L(\frac{7}{10},0)$, for which the irreducible modules are parametrized by the lowest conformal weight $h=0$, $3/80$, $1/10$, $7/16$, $3/5$, $2/3$. Hence the spectrum of the idempotent is determined as

$$\begin{aligned}&d(0)=51054,\ d(\tfrac{3}{80})=91392,\ d(\tfrac{1}{10})=47634,\ d(\tfrac{7}{16})=4864,\\&d(\tfrac{3}{5})=1938,\ d(\tfrac{3}{2})=1,d(2)=1.\end{aligned}$$

The map

$$\begin{aligned} 1 \quad &\text{on} \quad B^\natural(0) \oplus B^\natural(\tfrac{1}{10}) \oplus B^\natural(\tfrac{3}{2}) \oplus B^\natural(\tfrac{3}{5}) \oplus B^\natural(2), \\ -1 \quad &\text{on} \quad B^\natural(\tfrac{3}{80}) \oplus B^\natural(\tfrac{7}{16}), \end{aligned}$$

gives rise to an automorphism of $B^\natural$ by [Mi1]. Since the trace is equal to $51054 - 91392 + 47634 - 4864 + 1938 + 1 + 1 = 4372$, this map is identified with a 2A involution of the Monster.

W_3 algebra at $c = 4/5$ and 3A element

Let[2] $U = W_3(\frac{4}{5})$ be the vacuum sector of the W_3 algebra [Za] at $c = 4/5$. It is isomorphic to $L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ as modules over the Virasoro algebra. A realization of $W_3(\frac{4}{5})$ as a VOA as well as its representation theory are described in [KMY] and [Mi4].

There are 6 irreducible modules for this VOA, which are labeled as $(h, \sigma) = (0, 0)$, $(2/5, 0)$, $(2/3, \pm)$, $(1/15, \pm)$, where h is the lowest conformal weight and σ is the sign of the eigenvalue of the action of a certain primary vector $w \in W_3(\frac{4}{5})$ of conformal weight 3. Hence the spectrum is given by

$$d(0) = 57478,\ d(\tfrac{1}{15}) = 129168,\ d(\tfrac{2}{5}) = 8671,\ d(\tfrac{2}{3}) = 1566,\ d(2) = 1.$$

Consider the case when the primary vector w is also mapped to a vector in the real form $V^\natural_{\mathbb{R}}$. Then since $[w_{(2)}, t_{(1)}] = 0$ and $w_{(2)}$ is alternating with respect to the form $(\ |\ )$, we have a decomposition

$$B^\natural(\tfrac{1}{15}) = B^\natural(\tfrac{1}{15}, +) \oplus B^\natural(\tfrac{1}{15}, -), \quad B^\natural(\tfrac{2}{3}) = B^\natural(\tfrac{2}{3}, +) \oplus B^\natural(\tfrac{2}{3}, -),$$

into the sum of subspaces of equal dimensions for $h = 1/15$ and $2/3$. Now the map

$$1 \quad \text{on} \quad B^\natural(0) \oplus B^\natural(\tfrac{2}{5}) \oplus B^\natural(2), \qquad \zeta^{\pm 1} \quad \text{on} \quad B^\natural(\tfrac{2}{3}, \pm) \oplus B^\natural(\tfrac{1}{15}, \pm),$$

where ζ is a primitive 3rd root of unity, gives rise to an automorphism of $B^\natural$ by [Mi4]. Since the trace is equal to $57478 + 8671 + (\zeta + \zeta^{-1})(129168 + 1566)/2 + 1 = 783$, it is identified with a 3A element of the Monster.

This eigenspace decomposition is described in (24) of [MeN] and Lemma 4 of [No].

W_4 algebra at $c = 1$ and 4A element

Let $U = W_4(1)$ be the vacuum sector of the W_4 algebra at $c = 1$. It is realized as the fixed-point subspace V_L^+ of the lattice VOA V_L corresponding to the rank one lattice $L = \mathbb{Z}\gamma$ with $\langle \gamma, \gamma \rangle = 6$ with respect to the -1 automorphism of the lattice. It is generated by the conformal vector t and certain primary vectors w, z of conformal weight 3 and 4 respectively. We may use the representation theory of V_L^+ developed in [DN] and [Ab].

[2] The author presently does not know the existence of an embedding of $W_3(\frac{4}{5})$ into $V^\natural$.

There are 10 irreducible modules for this VOA, which are labeled as $(h, \sigma) = (0, 0)$, $(1, 0)$, $(1/12, 0)$, $(1/3, 0)$, $(3/4, \pm)$, $(1/16, \pm)$, $(9/16, \pm)$. Hence the spectrum is given by

$$\begin{aligned} &d(0) = 38226,\ d(\tfrac{1}{16}) = 94208,\ d(\tfrac{1}{12}) = 48600,\ d(\tfrac{1}{3}) = 11178, \\ &d(\tfrac{9}{16}) = 4096,\ d(\tfrac{3}{4}) = 552,\ d(1) = 23,\ d(2) = 1. \end{aligned}$$

These dimensions are determined as unique nonnegative integers that satisfy the formula (4.1), although the number of unknown dimensions exceeds the number of equations.

Suppose that the primary vectors w, z are mapped to the real form $V^\natural_{\mathbb{R}}$. Then the map

$$\begin{aligned} 1 \quad &\text{on} \quad B^\natural(0) \oplus B^\natural(\tfrac{1}{3}) \oplus B^\natural(1) \oplus B^\natural(2), \\ -1 \quad &\text{on} \quad B^\natural(\tfrac{1}{12}) \oplus B^\natural(\tfrac{3}{4}), \\ \pm\sqrt{-1} \quad &\text{on} \quad B^\natural(\tfrac{1}{16}, \pm) \oplus B^\natural(\tfrac{9}{16}, \pm) \end{aligned}$$

gives rise to an automorphism of $B^\natural$ by the fusion rules of $W_4(1)$. Since the trace is equal to $38226 - 48600 + 11178 - 552 + 23 + 1 = 276$, it is identified with a 4A element of the Monster. Note that the trace of the square of this map, i.e., of the map

$$\begin{aligned} 1 \quad &\text{on} \quad B^\natural(0) \oplus B^\natural(\tfrac{1}{12}) \oplus B^\natural(\tfrac{1}{3}) \oplus B^\natural(\tfrac{3}{4}) \oplus B^\natural(1) \oplus B^\natural(2), \\ -1 \quad &\text{on} \quad B^\natural(\tfrac{1}{16}) \oplus B^\natural(\tfrac{9}{16}), \end{aligned}$$

is equal to 276. Hence this map is identified with a 2B involution of the Monster as expected.

This eigenspace decomposition is described in Lemma 5 of [No][3].

W_5 algebra at $c = 8/7$ and 5A element

Unfortunately, the classification of irreducible modules and the determination of fusion rules based on the theory of VOA for $W_5(\frac{8}{7})$ seem to be missing. However, formally applying the expected properties of this algebra to our situation, the eigenspace decomposition is expected to be

$$\begin{aligned} &d(0) = 27228,\ d(\tfrac{2}{35}) = 72010,\ d(\tfrac{3}{35}) = 76912,\ d(\tfrac{17}{35}) = 6688,\ d(\tfrac{23}{35}) = 1520, \\ &d(\tfrac{2}{7}) = 12122,\ d(\tfrac{6}{7}) = 133,\ d(\tfrac{4}{5}) = 268,\ d(\tfrac{6}{5}) = 2,\ d(2) = 1. \end{aligned}$$

Since the trace of the map

$$\begin{aligned} 1 \quad &\text{on} \quad B^\natural(0) \oplus B^\natural(\tfrac{2}{7}) \oplus B^\natural(\tfrac{6}{7}) \oplus B^\natural(1) \oplus B^\natural(2), \\ \zeta^{\pm 1} \quad &\text{on} \quad B^\natural(\tfrac{2}{35}, \pm) \oplus B^\natural(\tfrac{17}{35}, \pm) \oplus B^\natural(\tfrac{6}{5}, \pm), \\ \zeta^{\pm 2} \quad &\text{on} \quad B^\natural(\tfrac{3}{35}, \pm) \oplus B^\natural(\tfrac{23}{35}, \pm) \oplus B^\natural(\tfrac{4}{5}, \pm), \end{aligned}$$

[3] There $24104 + 24104$ should read $47104 + 47104$.

where ζ is a primitive 5th root of unity, is equal to $27228+(\zeta+\zeta^{-1})(72010+6688+2)/2+(\zeta^2+\zeta^{-2})(76912+1520+268)/2+12122+133+1=134$, this map would be identified with a 5A element of the Monster (if we appropriately choose the signs above[4]).

Note 4.4 The fusion rules of W_n algebras constructed by the quantized Drinfeld-Sokolov reduction are determined by Frenkel et al. [FKW] via the Verlinde formula. In particular, the fusion ring of the first unitary series $W_n(c_n)$, $c_n = 2(n-1)/(n+2)$, is isomorphic to that of the level 2 integrable highest weight representations of the affine Kac-Moody Lie algebra of type $A^{(1)}_{n-1}$. Then the map $[\lambda] \mapsto \zeta_\lambda[\lambda]$, where $\zeta_\lambda = \exp(2\pi\sqrt{-1}\sum_{i=1}^{n-1} im_i/n)$ for a level 2 weight $\lambda = \sum_{i=1}^{n-1} m_i\bar{\Lambda}_i$, gives an automorphism of the fusion ring over $\mathbb{C}$. Therefore, we expect that an element of order n of the Monster would be obtained by using the decomposition of $V^\natural$ into the sum of irreducible $W_n(c_n)$-modules.

5 Generalization to Higher Degree

Recall the notations and assumptions in section 2.3. In particular, $\mathrm{Aut}\,V$ is supposed to be finite.

5.1 The trace functions

Let us set

$$o(u) = u_{(n-1)} : V \to V \tag{5.1}$$

for any $u \in V^n$ after Frenkel and Zhu [FrZh]. Consider the trace functions

$$\mathrm{Tr}\, o(a)q^{L_0} \quad \text{and} \quad \mathrm{Tr}\, o(a)o(b)q^{L_0}$$

for elements $a, b \in B$, where Tr denotes the trace over the whole space V.

In this subsection, we will express these trace functions, under the corresponding assumptions, in terms of the character

$$\mathrm{ch}\, V = \sum_{n=0}^{\infty} (\dim V^n) q^n \tag{5.2}$$

and the Eisenstein series $E_{2k} = E_{2k}(q)$, which we normalize as in [DM]:

$$E_{2k}(q) = -\frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!}\sum_{n=1}^{\infty}\left(\sum_{d|n} d^{2k-1}\right) q^n. \tag{5.3}$$

Here B_m are the Bernoulli numbers defined by

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

[4]This ambiguity is a matter of identification of the representations. There is no ambiguity if we adopt the labeling as in [FKW].

In particular,

$$E_2 = -\frac{1}{12} + 2q + 6q^2 + 8q^3 + 14q^4 + 12q^5 + \cdots$$
$$E_4 = \frac{1}{720} + \frac{1}{3}q + 3q^2 + \frac{28}{3}q^3 + \frac{73}{3}q^4 + 42q^5 + \cdots \tag{5.4}$$

The result is summarized in the following theorem

Theorem 5.1 *Let B be the Griess algebra of a VOA V such that $\operatorname{Aut} V$ is finite.*
(1) *If V is of class $\mathcal{S}^2$ then, for any $a \in B$,*

$$\operatorname{Tr} o(a) q^{L_0} = \frac{2(a|\omega)}{c} q \frac{d}{dq} \operatorname{ch} V.$$

(2) *If V is of class $\mathcal{S}^4$ then, for any $a, b \in B$,*

$$\begin{aligned}\operatorname{Tr} o(a)o(b) q^{L_0} = \bigg(& \frac{44(a|b) + 20(a|\omega)(b|\omega)}{c(5c+22)} \Big(q\frac{d}{dq} \Big)^2 \\ & -(11 + 60E_2) \frac{c(a|b) - 2(a|\omega)(b|\omega)}{3c(5c+22)} q \frac{d}{dq} \\ & +(11 + 120E_2 - 720E_4) \frac{c(a|b) - 2(a|\omega)(b|\omega)}{360(5c+22)} \bigg) \operatorname{ch} V.\end{aligned}$$

For instance,

$$\begin{aligned} & \operatorname{Tr}|_{V^3} o(a)o(b) \\ & \quad = \frac{-2(20c^2 + 40c \dim V^2 + (3c - 198)\dim V^3)}{c(5c+22)} (a|b) \\ & \quad + \frac{16(5c + 10\dim V^2 + 12\dim V^3)}{c(5c+22)} (a|\omega)(b|\omega), \\ & \operatorname{Tr}|_{V^4} o(a)o(b) \\ & \quad = \frac{-2(55c^2 + 98c\dim V^2 + 60c\dim V^3 + (4c - 352)\dim V^4)}{c(5c+22)} (a|b) \\ & \quad + \frac{4(55c + (5c+120)\dim V^2 + 60\dim V^3 + 84\dim V^4)}{c(5c+22)} (a|\omega)(b|\omega). \end{aligned}$$

We sketch the derivation of the formulae in the rest of this subsection. The formula (1) immediately follows from (2.14) and Lemma 2.8: if V is of class $\mathcal{S}^2$ then

$$\operatorname{Tr} a_{(1)} q^{L_0} = \frac{2(a|\omega)}{c} \operatorname{Tr} \omega_{(1)} q^{L_0} = \frac{2(a|\omega)}{c} \operatorname{Tr} L_0 q^{L_0} = \frac{2(a|\omega)}{c} q \frac{d}{dq} \operatorname{ch} V.$$

for any $a \in B$.

Suppose that V is of class $\mathcal{S}^4$ and consider two vectors $a, b \in B$ of the Griess algebra. Then, by Proposition 4.3.5 of [Zh], we have

$$\operatorname{Tr} o(a)o(b)q^{L_0} = \operatorname{Tr} o(a_{[-1]}b)q^{L_0} - E_2 \operatorname{Tr} o(a_{[1]}b)q^{L_0} - E_4 \operatorname{Tr} o(a_{[3]}b)q^{L_0}. \tag{5.5}$$

Here the operations $(a, b) \mapsto a_{[n]}b$, $(n \in \mathbb{Z})$, are another VOA structure on V introduced by Zhu [Zh], which is normalized so that

$$a_{[-1]}b = a_{(-1)}b + \frac{3}{2}a_{(0)}b + \frac{5}{12}a_{(1)}b + \frac{11}{720}a_{(3)}b,\ a_{[1]}b = a_{(1)}b - \frac{1}{6}a_{(3)}b,\ a_{[3]}b = a_{(3)}b. \tag{5.6}$$

for $a, b \in B$. Then, by (5.5) and (2.15), we have

$$\begin{aligned}\operatorname{Tr}\ o(a)o(b)q^{L_0} &= \frac{11}{720}\operatorname{Tr} o(\mathbf{1})q^{L_0} + \frac{5(a|b)}{3c}\operatorname{Tr} o([2])q^{L_0} + \frac{3(a|b)}{c}\operatorname{Tr} o([3])q^{L_0} \\ &+ \frac{6c(a|b) - 12(a|\omega)(b|\omega)}{c(5c+22)}\operatorname{Tr} o([4])q^{L_0} + \frac{44(a|b) + 20(a|\omega)(b|\omega)}{c(5c+22)}\operatorname{Tr} o([2,2])q^{L_0} \\ &- E_2\frac{4(a|b)}{c}\operatorname{Tr} o([2])q^{L_0} + \frac{1}{6}E_2(a|b)\operatorname{Tr} o(\mathbf{1})q^{L_0} - E_4(a|b)\operatorname{Tr} o(\mathbf{1})q^{L_0}.\end{aligned} \tag{5.7}$$

Hence we get Theorem 5.1 (2) by using the following:

$$\operatorname{Tr} o([2]) = q\frac{d}{dq}\operatorname{ch} V,\ \operatorname{Tr} o([3]) = -2q\frac{d}{dq}\operatorname{ch} V,\ \operatorname{Tr} o([4]) = 3q\frac{d}{dq}\operatorname{ch} V, \tag{5.8}$$

$$\operatorname{Tr} o([2,2])q^{L_0} = \left(\left(q\frac{d}{dq}\right)^2 + \left(\frac{13}{6} + 2E_2\right)q\frac{d}{dq} - \left(\frac{11c}{1440} + \frac{c}{12}E_2 - \frac{c}{2}E_4\right)\right)\operatorname{ch} V.$$

Here only the last one is not obvious. By (5.5), we have

$$\operatorname{Tr} o(\omega)o(\omega)q^{L_0} = \operatorname{Tr} o(\omega_{[-1]}\omega)q^{L_0} - E_2\operatorname{Tr} o(\omega_{[1]}\omega)q^{L_0} - E_4\operatorname{Tr} o(\omega_{[3]}\omega)q^{L_0}. \tag{5.9}$$

Substituting

$$o(\omega_{[-1]}\omega) = [2,2]_{(3)} - \frac{13}{6}[2]_{(1)} + \frac{11}{144}c,\ o(\omega_{[1]}\omega) = 2[2]_{(1)} - \frac{c}{12},\ o(\omega_{[3]}\omega) = \frac{c}{2}.$$

and

$$\operatorname{Tr} o(\omega)q^{L_0} = \operatorname{Tr} L_0 q^{L_0} = q\frac{d}{dq}\operatorname{ch} V,\ \operatorname{Tr} o(\omega)o(\omega)q^{L_0} = \left(q\frac{d}{dq}\right)^2\operatorname{ch} V,$$

we have the result.

5.2 McKay-Thompson series for 2A involution

Let V be the moonshine module $V^\natural$. In this subsection, we will show that the McKay-Thompson series for a 2A involution is determined by the formulae above using the fact that the character of $V^\natural$ is given by

$$\begin{aligned}\operatorname{ch} V^\natural &= q(J(q) - 744) \\ &= 1 + 196884q^2 + 21493760q^3 + 864299970q^4 + 20245856256q^5 + \cdots\end{aligned} \tag{5.10}$$

and that the characters of $L(\frac{1}{2},h)$ are given by

$$
\begin{aligned}
\chi_0(q) &= \mathrm{ch}\, L(\tfrac{1}{2},0) = \frac{1}{2}\left(\prod_{k=0}^{\infty}(1+q^{k+1/2}) + \prod_{k=0}^{\infty}(1-q^{k+1/2})\right),\\
\chi_{1/2}(q) &= \mathrm{ch}\, L(\tfrac{1}{2},\tfrac{1}{2}) = \frac{1}{2}\left(\prod_{k=0}^{\infty}(1+q^{k+1/2}) - \prod_{k=0}^{\infty}(1-q^{k+1/2})\right),\\
\chi_{1/16}(q) &= \mathrm{ch}\, L(\tfrac{1}{2},\tfrac{1}{16}) = q^{1/16}\prod_{k=1}^{\infty}(1+q^{k}).
\end{aligned}
\tag{5.11}
$$

Now, let e be an idempotent of central charge $1/2$ in the Conway-Griess algebra $B^\natural$ and consider the corresponding Virasoro action $L^e_n = e_{(n+1)}$. Consider the subspace

$$P(h) = \{v \in V \mid L^e_n v = 0 \ \text{if}\ n \geq 1 \ \text{ and }\ L^e_0 v = hv\} \tag{5.12}$$

for each $h = 0, 1/2, 1/16$, and set

$$z_h(q) = \sum_{n=0}^{\infty} \dim(V^n \cap P(h))q^n. \tag{5.13}$$

Then we have

$$
\begin{aligned}
z_0(q)\chi_0(q) + q^{-1/2}z_{1/2}(q)\chi_{1/2}(q) + q^{-1/16}z_{1/16}(q)\chi_{1/16}(q) &= \mathrm{Tr}\, q^{L_0},\\
z_0(q)\dot{\chi}_0(q) + q^{-1/2}z_{1/2}(q)\dot{\chi}_{1/2}(q) + q^{-1/16}z_{1/16}(q)\dot{\chi}_{1/16}(q) &= \mathrm{Tr}\, o(e)q^{L_0},\\
z_0(q)\ddot{\chi}_0(q) + q^{-1/2}z_{1/2}(q)\ddot{\chi}_{1/2}(q) + q^{-1/16}z_{1/16}(q)\ddot{\chi}_{1/16}(q) &= \mathrm{Tr}\, o(e)^2q^{L_0},
\end{aligned}
\tag{5.14}
$$

where $\dot{\chi}(q) = qd/dq\chi(q)$ and $\ddot{\chi}(q) = (qd/dq)^2\chi(q)$. By Theorem 5.1,

$$\mathrm{Tr}\, q^{L_0} = \mathrm{ch}\, V^\natural, \quad \mathrm{Tr}\, o(e)q^{L_0} = \frac{1}{48}q\frac{d}{dq}\mathrm{ch}\, V^\natural, \tag{5.15}$$

$$\mathrm{Tr}\, o(e)^2q^{L_0} = \left(\frac{49}{13632}\Big(q\frac{d}{dq}\Big)^2 - \frac{47(11+60E_2)}{81792}q\frac{d}{dq} + \frac{47(11+120E_2-720E_4)}{163584}\right)\mathrm{ch}\, V^\natural,$$

where $\mathrm{ch}\, V^\natural$ is given by (5.10). Therefore, the condition (5.14) determines the series $z_0(q)$, $z_{1/2}(q)$ and $z_{1/16}(q)$, so the McKay-Thompson series $T_{2\mathrm{A}}(q)$ via

$$T_{2\mathrm{A}}(q) = q^{-1}\Big(z_0(q)\chi_0(q) + q^{-1/2}z_{1/2}(q)\chi_{1/2}(q) - q^{-1/16}z_{1/16}(q)\chi_{1/16}(q)\Big). \tag{5.16}$$

The result is written as a rational expression involving the functions $J(q)$, $\chi_h(q)$, their first and the second derivatives and the Eisenstein series $E_2(q)$ and $E_4(q)$. We do not include the explicit form in this paper.

Appendix

A.1 The coefficients in the trace formula (4)

$$A_1 = -c(2100c^5 + 53650c^4 + 304049c^3 - 980942c^2 - 1641936c + 229152)$$
$$+(2455c^4 - 193958c^3 + 4032472c^2 + 539488c - 1651584)d,$$
$$A_2 = -c(1050c^5 + 30965c^4 + 279826c^3 + 609848c^2 - 271248c - 150144)$$
$$-c(60c^4 - 4929c^3 + 96248c^2 + 258428c - 56304)d,$$
$$A_3 = -c(1050c^5 + 31085c^4 + 270928c^3 + 726848c^2 + 1748472c - 79008)$$
$$+c(60c^4 - 3969c^3 + 20752c^2 + 1761292c + 127440)d,$$
$$B = 4c(1050c^4 + 30905c^3 + 289750c^2 + 281168c$$
$$-4d(120c^4 - 14853c^3 + 424928c^2 + 11132c - 206448)),$$
$$C = 8c(d-1)(120c^3 - 9437c^2 + 187858c + 22968),$$
$$D = -192c(d-1)(100c^2 - 4297c - 2852)(d-1), \quad E = 15744c(30c + 47).$$

A.2 The coefficients in the trace formula (5)

$$A_{1,2,3,4,5} = -5c(46200c^6 + 2154600c^5 + 31531073c^4$$
$$+123663366c^3 - 560461448c^2 - 1390398720c - 168205824)$$
$$-5d(100c^6 - 2405c^5 - 1037398c^4 + 70463896c^3 - 1249353984c^2 + 60544768c + 766334976),$$
$$A_{1,2,4,3,5}$$
$$= c(1500c^5 - 161985c^4 + 5500754c^3 - 19601928c^2 - 1338547904c - 3497905152)(d-1),$$
$$A_{1,2,5,3,4}$$
$$= c(-1500c^5 + 147745c^4 - 3380778c^3 - 83375368c^2 + 2968841472c + 3711048192)(d-1),$$
$$A_{1,3,2,4,5} = -c(115500c^6 + 5849050c^5 + 102135165c^4$$
$$+720684894c^3 + 1549368552c^2 - 664210624c + 4461754368)$$
$$+cd(300c^5 - 81505c^4 + 5253294\ c^3 - 87363968c^2 - 611758944c + 4940713728),$$
$$A_{1,3,4,2,5}$$
$$= c(500c^5 - 29035c^4 + 518574c^3 - 15730088c^2 + 553755136c - 3442893312)(d-1),$$
$$A_{1,3,5,2,4}$$
$$= c(-500c^5 + 14795c^4 + 1601402c^3 - 87247208c^2 + 1076538432c + 3656036352)(d-1),$$
$$A_{1,4,2,3,5} = -c(115500c^6 + 5848050c^5 + 102268115c^4$$
$$+715702714c^3 + 1553240392c^2 + 1228092416c + 4516766208)$$
$$-cd(700c^5 - 51445c^4 - 271114c^3 + 83492128c^2 - 1280544096c - 4995725568),$$
$$A_{1,4,3,2,5}$$
$$= c(-500c^5 + 82955c^4 - 5023222c^3 + 122369272c^2 - 861258816c - 1610972160)(d-1),$$
$$A_{1,4,5,2,3}$$
$$= c(500c^5 - 118155c^4 + 6583582c^3 - 91119048c^2 - 815764608c + 3601024512)(d-1),$$

$$
\begin{aligned}
A_{1,5,2,3,4} = -c(&115500c^6 + 5847050c^5 + 102401065c^4 \\
&\qquad +710720534c^3 + 1557112232c^2 + 3120395456c + 4571778048) \\
&-cd(1700c^5 - 184395c^4 + 4711066c^3 + 79620288c^2 - 3172847136c - 5050737408),
\end{aligned}
$$

$$
\begin{aligned}
&A_{1,5,3,2,4} \\
&\quad = c(500c^5 - 49995c^4 - 41042c^3 + 118497432c^2 - 2753561856c - 1665984000)(d-1), \\
&A_{1,5,4,2,3} \\
&\quad = c(-500c^5 + 103915c^4 - 4463606c^3 - 11858248c^2 + 2446058176c - 3387881472)(d-1), \\
&A_{2,3,1,4,5} \\
&\quad = -3c(100c^5 - 21675c^4 + 907054c^3 + 11023128c^2 - 806389760c + 1100745216)(d-1), \\
&A_{2,4,1,3,5} \\
&\quad = c(700c^5 - 67925c^4 + 2261018c^3 - 36941224c^2 + 526866240c - 3357247488)(d-1), \\
&A_{2,5,1,3,4} \\
&\quad = c(1700c^5 - 200875c^4 + 7243198c^3 - 40813064c^2 - 1365436800c - 3412259328)(d-1),
\end{aligned}
$$

$$
\begin{aligned}
B_1 &= 4c(115500c^5 + 5848250c^4 + 101927925c^3 + 740910478c^2 + 1067413032c + 217343424) \\
&\qquad +4d(500c^5 + 288745c^4 - 25478878c^3 + 569319488c^2 - 269795104c - 478959360), \\
B_2 &= -4c(8100c^4 - 616655c^3 + 8745246c^2 + 142937384c - 614801472)(d-1), \\
B_3 &= 4c(8100c^4 - 482575c^3 - 1572066c^2 + 339056296c - 368532288)(d-1), \\
C &= -8c(1780c^4 - 264997c^3 + 12872162c^2 - 203786696c - 26642880)(d-1), \\
D &= 64c(3620c^3 - 510813c^2 + 15237868c + 4458096)(d-1), \\
E &= 256c(2095c^3 - 161208c^2 + 3064358c + 3847956)\ (d-1), \\
F &= -3840c(3000c^2 - 125177c - 223532)(d-1), \quad G = 333312c(90c+259)(d-1), \\
H &= -\frac{c}{12}(100c^5 - 13295c^4 + 498218c^3 - 387184\ c^2 - 189230304c - 5501184)(d-1).
\end{aligned}
$$

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